14.5 Chain Rule

It requires range$(g) \subset$ Domain of f and f differentiable in D. and g differentiable in its domain.

1-D. $y = f(x)$, $x = g(t)$

$\Rightarrow$ $(f \circ g)(t) = f(g(t))$

$$\frac{d}{dt}[f \circ g](t) = f'(g(t)) \cdot g'(t).$$

It's also written as

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

2-D $z = f(x,y)$ and $x = g(t)$, $y = h(t)$

1st CASE Then, there is a new function

Composition: $F(t) = f(g(t), h(t))$

and $\frac{dz}{dt}(t) = \frac{dF}{dt}(t) = \frac{\partial f}{\partial x}(g(t))\,\frac{dg}{dt}(t) + \frac{\partial f}{\partial y}(h(t))\,\frac{dh}{dt}(t)$

It's also written as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Application :

14.5 #2) (7th Edition) $Z = \cos(x+4y)$, $x = 5t^4$, $y = 1/t$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

$$\boxed{\frac{dz}{dt} = -\sin(x+4y)\,20t^3 - 4\sin(x+4y)\left(-\frac{1}{t^2}\right)}$$

or

omit

$$\frac{dz}{dt} = -\sin\left(5t^4+\frac{4}{t}\right)20t^3 + \frac{4}{t^2}\sin\left(5t^4+\frac{4}{t}\right)$$

$$= \left(4/t^2 - 20t^3\right)\sin\left(5t^4+\frac{4}{t}\right)$$

or $\boxed{\frac{dz}{dt} = \left(\frac{4}{t^2} - 20t^3\right)\sin(x+4y).}$

If question is find $\frac{dz}{dt}$ when $\boxed{t=2}$

then, $\frac{dz}{dt} = (1-160)\sin(80+\overset{2}{\cancel{4}/2}) = -159\sin(82).$

2-D 2nd CASE $Z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$

Composition: $F(s,t) = f(g(s,t), h(s,t))$

and $\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}(g(s,t),h(s,t))\frac{\partial g}{\partial s}(s,t) + \frac{\partial f}{\partial y}(g(s,t),h(s,t))\frac{\partial h}{\partial s}(s,t)$

$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial x}(\,''\,)\frac{\partial g}{\partial t}(s,t) + \frac{\partial f}{\partial y}(\,''\,)\frac{\partial h}{\partial t}(s,t).$

or

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial t}$$

Application: 14.5/#10) (7th edition) $z = e^{x+2y}$, $x = s/t$, $y = s/t$

$$\frac{\partial z}{\partial s} = e^{x+2y}\,\frac{1}{t} + 2\,e^{x+2y}\,\frac{1}{t} = \frac{3}{t}e^{x+2y}$$

$$\frac{\partial z}{\partial t} = e^{x+2y}\left(-\frac{s}{t^2}\right) + 2\,e^{x+2y}\left(-\frac{s}{t^2}\right) = -\frac{3s}{t^2}\,e^{x+2y}$$

If question is "find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at $s=2$, $t=1$."

Then,

$$\frac{\partial z}{\partial s} = 3\,e^{2+4} = 3e^{6}$$

$$\frac{\partial z}{\partial t} = -6\,e^{6}.$$

The chain rule can be extended to more Variables. In fact,

$w = W(x, y, z)$. differentiable function of 3 Variables and each x, y, z are differentiable functions of 2 variables (It may be any number of Variables m)

$x = x(u, v)$, $y = y(u, v)$, $z(u, v)$.

Then

$w(u, v) = W(x(u, v), y(u, v), z(u, v))$

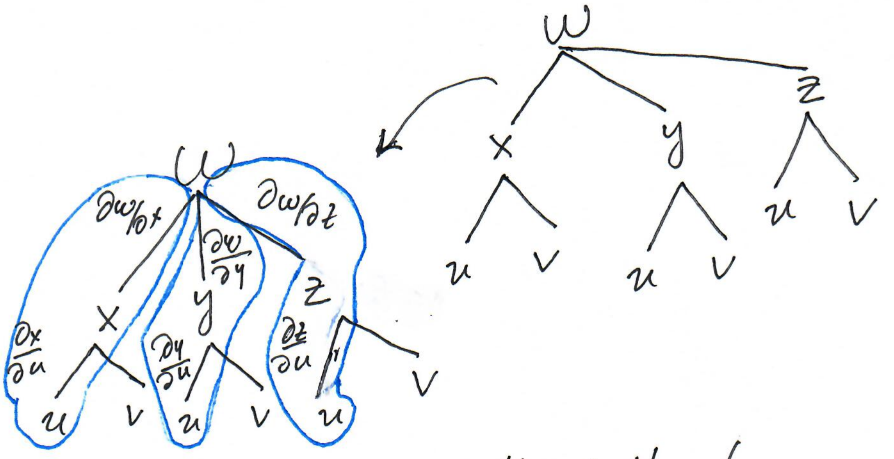
and w is also differentiable as a function of u and v.

More precisely,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Tree diagram:



To find $\frac{\partial w}{\partial u}$, we consider all the paths from $w \xrightarrow{to} u$ and form products of partial derivatives along each path and finally add all of them.

Second Derivatives and the chain Rule.

Given

$$z = z(x,y), \quad x = x(r,\theta)$$
$$y = y(r,\theta)$$
$$\downarrow$$
$$z = z(r,\theta) = z(x(r,\theta), y(r,\theta))$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

Find $\frac{\partial^2 z}{\partial r^2}$.

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)$$

or

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)$$

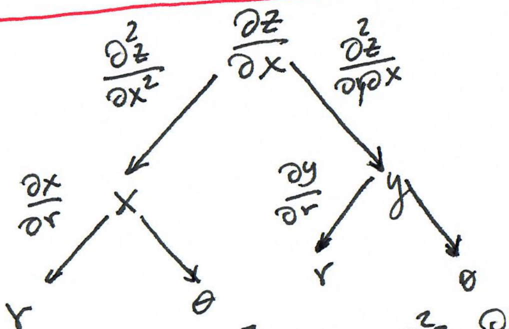
or

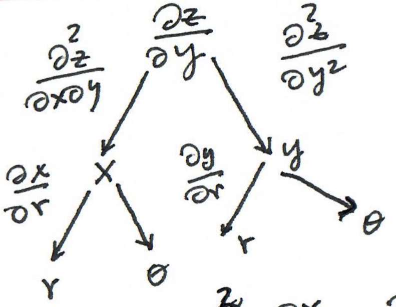
$$\frac{\partial^2 z}{\partial r^2} \overset{\text{Add!}}{=} \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)$$

or

$$\frac{\partial^2 z}{\partial r^2} \overset{\text{prod. rule}}{=} \boxed{\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial r^2} + \boxed{\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial r^2}$$

Now, notice that





Then, $\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial r}$ $\quad \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial r}\right)^2 + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial r^2}$$
$$+ \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial r}\right)^2 + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial r^2}.$$

Finally,

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial r}\right)^2 + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial r}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r}\frac{\partial y}{\partial r}$$
$$+ \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial r^2} + \frac{\partial z}{\partial x}\left(\frac{\partial^2 x}{\partial r^2}\right)$$

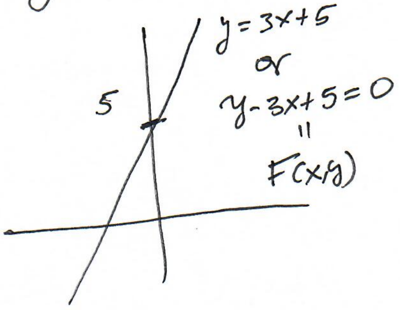
Implicit Differentiation

Consider the equation

$$y - 3x - 5 = 0 \qquad (1)$$

Solving for y $\quad y = 3x + 5 \qquad (2)$

So we can say that Equ. (1) defines $y = f(x)$



Equ. (1) can also be written as

$F(x,y) = 0$, where

$F(x,y) = y - 3x - 5$, clearly from (2) $\boxed{\frac{dy}{dx} = 3}$

Implicit differentiation assuming x independent variable and y dependent on x; y(x).

$$G(x) = F\left(\overset{=x}{x(x)}, y(x)\right)$$

$$\frac{dG}{dx} = \frac{\partial F}{\partial x}\overset{=1}{\frac{dx}{dx}} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$\boxed{\text{Now, if } F(x,y) = 0,}$ differentiating both sides

$$\frac{\partial F}{\partial x}\overset{=1}{\frac{dx}{dx}} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ assumig $\frac{\partial F}{\partial y} \neq 0.$

$$\frac{dy(x)}{dx} = -\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)} = -\frac{F_x}{F_y}(x,y)$$

Application to $F(x,y) = y-3x-5=0$

$$\frac{dy(x)}{dx} = -\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)} = -\frac{-3}{1} = 3$$

In general, this is also true under appropriate conditions.

Implicit Function Theorem

If
- F defined in $B_r(a,b)$ 

- $F(a,b)=0$
- $F_y(a,b) \neq 0$
- F_x, and F_y are conts on $B_r(a,b)$

Then $F(x,y)=0$ defines $y=f(x)$ implicitly near (a,b)

and

$$\frac{dy(x)}{dx} = -\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)}.$$

Example

14.5

#27) $y\cos x = x^2+y^2$, $x=0,\ y=1$

(0,1) is on the curve

Or $F(x,y) = y\cos x - x^2 - y^2 = 0$

Alternative:

$y\cos x - x^2 - y^2 = 0$

$dy/dx \cos x - y\sin x - 2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y\sin x + 2x}{\cos x - 2y}$

Find $\frac{dy}{dx}$.

answ: $\frac{dy}{dx}(x) = -\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)} = -\frac{-y\sin x - 2x}{\cos x - 2y} = \frac{y\sin x + 2x}{\cos x - 2y}.$

Find $\frac{dy}{dx}(0)_{y=1}$?

$\frac{dy}{dx}(0) = \frac{\frac{\partial F}{\partial x}(0,1)}{\frac{\partial F}{\partial y}(0,1)} = \frac{1(0)+2(0)}{1-2(1)} = 0$

This result can be extended to functions of more variables.

Suppose z is given implicitly as a fn. of x and y

For example,

#33) $e^z - xyz = 0$, Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

$F(x,y,z) = 0$, x, y indep. and $\overset{\text{dependent}}{z(x,y)}$ $F(\overset{=x}{x(x,y)}, \overset{=y}{y(x,y)}, z(x,y))$

Then $\frac{\partial F}{\partial x}\overset{=1}{\frac{\partial x}{\partial x}} + \frac{\partial F}{\partial y}\overset{=0}{\frac{\partial y}{\partial x}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$

$\Rightarrow \frac{\partial z}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y,z)}{\frac{\partial F}{\partial z}(x,y,z)}$, Similarly $\frac{\partial z}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y,z)}{\frac{\partial F}{\partial z}(x,y,z)}$.

Application:

#33) If $e^{z}-xyz=\overset{=F(x,y,z)}{0}$, Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$\frac{\partial z(x,y)}{\partial x} = -\frac{\frac{\partial F}{\partial x}(x,y,z)}{\frac{\partial F}{\partial z}(x,y,z)} = -\frac{-y\,z(x,y)}{e^{z(x,y)}-xy} = \frac{y\,z(x,y)}{e^{z(x,y)}-xy}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{-xz}{e^{z}xy} = \frac{xz}{e^{z}-xy}.$$

Find $\frac{\partial z}{\partial x}\left(\frac{e}{\sqrt{2}}, \frac{e}{\sqrt{2}}\right)$

(Knowing that the point $\left(\frac{e}{\sqrt{2}}, \frac{e}{\sqrt{2}}, 2\right)$ is on the Surface.
$e^{2}-\left(\frac{e}{\sqrt{2}}\right)\left(\frac{e}{\sqrt{2}}\right)2 = e^{2}-e^{2}=0$)

$$\frac{\partial z}{\partial x}\left(\frac{e}{\sqrt{2}}, \frac{e}{\sqrt{2}}\right) = \frac{2e/\sqrt{2}}{e^{2}-\frac{e^{2}}{2}} = \frac{\frac{2e}{\sqrt{2}}}{\frac{e^{2}}{2}} = \boxed{\frac{4e}{\sqrt{2}e^{2}}} = \frac{4\sqrt{2}}{2e}$$

or $\boxed{\frac{\partial z}{\partial x}\left(e/\sqrt{2}, e/\sqrt{2}\right) = 2\sqrt{2}e^{-1}}$